2024 AGMC Tianyi Summer Cup Math Competition

【Team Round】

Question Sheet

(Full Marks: 100 points　　Duration: 160 minutes)

You will have 10 minutes to read the following instructions and fill in your information.

Uploading Instruction

This competition uses the Rainclassroom for answering questions. It requires the team leader to use their Rainclassroom account to take photos and upload the following pages: [Team Information] on page 2, the Fill-in-the-blank Answer Sheet on page 3, and the blank A4 paper prepared for solving problems (only the team leader needs to upload these).

Types of questions

The competition consists of 12 questions with a total score of 100 points.

It contains four main sections: Geometry, Algebra, Number Theory, and Combinatorics. Each section is worth 25 points and includes one Fill-in-the-blank Question (2 points), one Short Answer Question (5 points), and one Theme Question (18 points).

Examination Instruction

The duration is 160 minutes. No breaks during the exam.

All participants can discuss the questions together. If participants are discussing offline, ensure all members are within the proctor's eyesight. If participant needs to discuss online, prepare two devices: one for the proctoring session on Tencent Meeting and another for group discussion on Tencent Meeting.

This competition is an online open-book exam, and consulting paper materials is allowed. The use of electronic devices such as computers, mobile phones, calculators, etc. is prohibited. Any use of electronic devices will result in disqualification.

Information Filling Instruction

Please fill in the following Team Information, followed by the information for Contestant 1, Contestant 2, and Contestant 3 in sequence. If there is a 4th player on the team, fill in the information for Contestant 4. If there is no 4th player, just write [None] in each blank of the information of Contestant 4.

If all team members are in the same school, fill in the school name in the [School] blank of the Team Information and write [Same as above] in the [School] blank of each contestant's personal information. If the team is composed of students from multiple schools, please write [Multiple Schools United] in the [School] blank of the Team Information and fill in each player's respective school name in the [School] blank of each contestant's personal information.

Photograph and upload this page

Team Information			
Name			
ID		**School**	
Contestant 1			
Name		**School**	
ID		**Phone**	
QQ		**Email**	
Contestant 2			
Name		**School**	
ID		**Phone**	
QQ		**Email**	
Contestant 3			
Name		**School**	
ID		**Phone**	
QQ		**Email**	
Contestant 4			
Name		**School**	
ID		**Phone**	
QQ		**Email**	

Photograph and upload this page

Fill-in-the-blank Answer Sheet			
Geometry		**Algebra**	
Number Theory		**Combinatorics**	

You will have 10 minutes to read the following instructions and fill in your information.

No information about questions is available with in the 10-minute period.

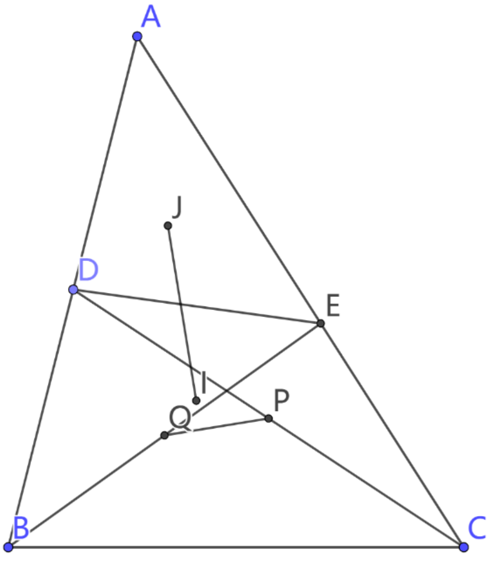
Section 1【Geometry】

1. Fill in the blank

Given a triangle with side lengths of 4, 5, and 6, the distance between the circumcenter and the incenter is__________ .

2. Short Response Question

In triangle $\triangle ABC$, points D and E are on AB and AC, respectively, with $BD = CE$. Points I and J are the incenters of $\triangle ABC$ and $\triangle ADE$, respectively. Points P and Q are the midpoints of CD and BE, respectively. Prove: $IJ \perp PQ$.



3. Theme Question

The rule of *harmonic range of points* is simple and elegant: For point C on segment AB and point D on the extension of segment AB, if $\frac{AC}{BC}=\frac{AD}{BD}$, then A, B, C, and D are called *harmonic range of points*.

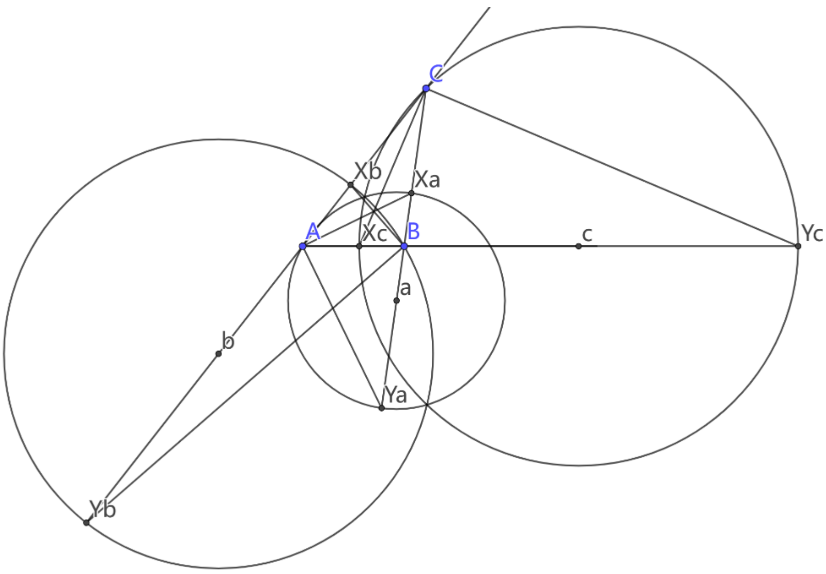
(1) [4] A, B, C, and D are harmonic range of points. Draw a circle with CD as its diameter, then any moving point P on this circle always satisfies $\frac{AP}{BP}=k\ (k\neq 1)$. Given a non-equilateral triangle $\triangle ABC$, let the internal and external angle bisectors of $\angle A$ intersect line BC at points X and Y, respectively. Draw a circle with XY as its diameter, known as the *Apollonius Circle*, and denote this circle as circle a. Similarly, construct circles b and c through points B and C. Prove that circles a, b, and c have exactly two common points.

(2) [8] Prove the following conclusion about harmonic range of points:
The projections of the three vertices of $\triangle ABC$ onto their corresponding opposite sides are D, E, and F, respectively, with the circumcenter O. The escenters corresponding to points B and C are I_b and I_c. Lines EF and OD intersect I_bI_c at points X and Y. Prove: I_b, X, Y, and I_c are harmonic range of point.

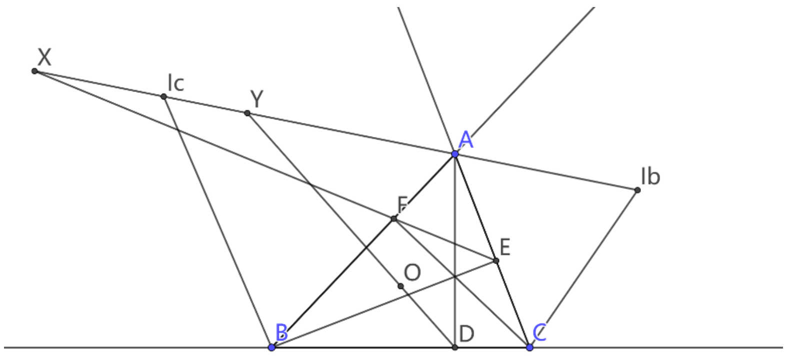
(3) **[6]** Under the conditions of (2), let the circle K passing through points B, Y, and C intersect I_bI_c at a second point T. There exists a point S in the plane such that $\triangle TBC\backsim\triangle SFE$, and this point can be constructed by drawing only one circle using a compass (all given conditions in the diagram can be used). Find this point and prove the conclusion.

[You can use these diagrams to solve the question]

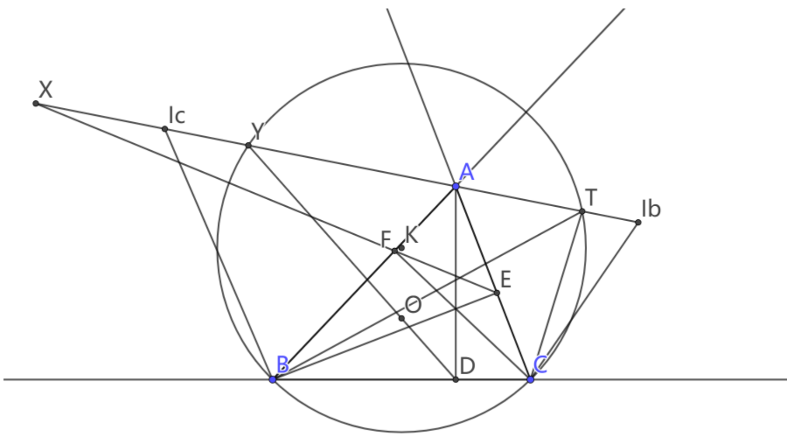
（1）



（2）



（3）



Section 2【Algebra】

1. Fill in the blank

Let $a, b, c \in \mathbb{C}$, $\begin{cases} a^2 = b - c \\ b^2 = c - a \\ c^2 = a - b \end{cases}$. Compute $a + b + c =$ __________ .

2. Short Response Question

Let x, y, z be real numbers and $xyz = 1$. There is a function f: $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$f(x)^2 - f(y)f(z) = x(x + y + z)(f(x) + f(y) + f(z))$$

Find $f(x)$.

3. Theme Question

We define $K[x]$ as the univariate polynomial ring over the field K. $\forall f(x) \in K[x]$, $f(x) = \sum_{i=0}^{n} a_i x^i$. n is called the degree of $f(x)$, denoted as $n = \deg f(x)$, $n \in \mathbb{N}$, $a_i \in K$. We say that $f(x)$ is reducible over K. When $\exists\, g(x)$, $h(x) \in K[x]$, the degree of $g(x)$ and $h(x)$ is not equal to 0, and $f(x) = g(x) \cdot h(x)$.

(1) [3] Proof: $x^5 - x - 1$ is not reducible over $Q[x]$. Furthermore, discuss for which values of $n \in \mathbb{N}^+$ such that $x^n - x - 1$ is not reducible over $Q[x]$.

(2) [3] Let p be a prime number, the base-b expansion of p is $\sum_{i=0}^{n} a_i b^i$, where a_i refers to each digit, $0 \le a_i \le b-1$. Proof: Polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is not reducible over $Q[x]$.

(3) [5] Let p be a prime number. There exist $f(x) \in Z[x]$, and $f(x)$ is not reducible over $Z[x]$. $[(-1)^{\deg f(x)} f(0)]^{\frac{1}{p}}$ is an irrational number. Proof: $f(x^p)$ is not reducible over $Z[x]$.

(4) [7] Let P and Q be two polynomials with complex coefficients. For any point Z on the circumference of a circle centered at the origin with radius r, $r \in R^+$. $|P(Z) - Q(Z)| < |Q(Z)|$ holds. Proof: P and Q have the same number of zeros inside this circle (counting multiplicities, i.e., $(Z - Z_0)^k$ counts as k zeros).

Section 3【Number Theory】

1. Fill in the blank

Let n be a two-digit number. If $2n+3 \mid 2^{n!}-1$, compute $n_{\min} =$ __________ .

2. Short Response Question

$v_p(n)$ denotes the exponent of the prime number p in the prime factorization of the positive integer n. Let $a \in \mathbb{N}^+$, there are at least 2 odd prime number p such that $\sum_{k=1}^{a}(-1)^{v_p(k!)} < 0$. Find $a_{\min}$.

3. Theme Question

Quadratic Residue is an important concept in elementary number theory, providing a foundational theoretical basis for determining whether quadratic (or even higher-order) congruences have solutions. In the analysis of quadratic residues, the introduction of the Legendre symbol not only translates all verbal language into mathematical language but also greatly simplifies calculations, making it easier to derive a series of properties of quadratic residues. Let's provide the definitions first:

Quadratic residues

Given $m \in N, m \geq 2, a \in Z,$ $\gcd(a, m) = 1$. If there exists $x \in Z$ such that $x^2 \equiv a \pmod{m}$, a is called the *Quadratic Residue* modulo m; otherwise, a is called the *Quadratic Non-residue* modulo m.

Legendre symbol

Give prime number p, $a \in Z$:

$$(\frac{a}{p}) = \begin{cases} 1 & a \text{ is the quadratic residue modulo } m \\ -1 & a \text{ is the quadratic non-residue modulo } m \\ 0 & p|a \end{cases}$$

Now we present two important theorems related to quadratic residues and the Legendre symbol, which can be directly used when solving this question.

Euler's criterion

Let p be an odd prime, $\gcd(a, p) = 1$.

If $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$, a is quadratic residue modulo m;

If $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$, a is quadratic non-residue modulo m.

Gauss's lemma

Let p be an odd prime, $\gcd(a, p) = 1$.

$$(\frac{x}{p}) = (-1)^{|\{1 \leq k < \frac{p}{2}: \{kx\}_p > \frac{p}{2}\}|}$$

where $\{kx\}_p$ refers to the remainder of kx modulo p.

Now let's prove the following conclusions:

(1) [3] Let p be an odd prime, $\gcd(a, p) = 1$, $f(x)$ is an integer-coefficient polynomial. As x runs through a complete residue system modulo p,

$$\sum_{x \pmod p} \left(\frac{ax+b}{p}\right) = 0$$

(2) [4] Under the conditions of (1) : If $f(x)$ is an odd function, then when p is a prime number of the form $4m + 1$:

$$\sum_{x=1}^{p-1} \left(\frac{f(x)}{p}\right) = 2\sum_{x=1}^{\frac{p-1}{2}} \left(\frac{f(x)}{p}\right)$$

when p is a prime number of the form $4m + 3$:

$$\sum_{x=1}^{p-1} \left(\frac{f(x)}{p}\right) = 0$$

(3) [5] When p is a sufficiently large prime number in the form of $4m + 3$

Proof: there must exist x, y, $z \in \{1, 2, \ldots, p - 1\}$, such that:

$$\begin{cases} \left(\frac{x}{p}\right) = \left(\frac{x+1}{p}\right) = 1 \\ -\left(\frac{y}{p}\right) = \left(\frac{y+1}{p}\right) = 1 \\ \left(\frac{z}{p}\right) = -\left(\frac{z+1}{p}\right) = 1 \end{cases}$$

(4) [6] Let p be an odd prime, $\gcd(x(1-x), p) = 1$, then:

$$(-1)^{\left|\{1 \le k < \frac{p}{2}: \{kx\}_p > k\}\right|} = \left(\frac{2x(1-x)}{p}\right)$$

Section 4【Combinatorics】

1. Fill in the blank

You participated in a game of *Russian Roulette*. The dealer loaded bullets into 3 random chambers of the magazine. The unfortunate person next to you fired two shots: the first shot was safe, but the second shot sent him to the heaven. The gun is then handed to you. You ________ [fill in 'should'/'should not'] spin the magazine (to a random chamber) before firing, and your survival probability is ________ **.** (The image below shows the gun used—a Colt Python, lethal with one shot, containing 6 chambers, with the cylinder rotating counterclockwise by one chamber after each shot.)"

2. Short Response Question

In a 100×100 grid, each row and each column contains 3 red squares, amounting to a total of 300 red squares. We can always remove k red squares such that no 2×2 grid is entirely red. Find $k_{\min}$.

3. Theme Question

Drawer Principle: For $a_i \in R$, $m = \sum_{i=1}^{n} a_i$, we have $\exists\, j,\ a_j \geq \frac{m}{n}$.

(1) [3] Let A be the subset of Z_n, $Z_n = \{0, 1, 2 \ldots n-1\}$, and $|A| \leq \frac{\ln n}{1.7}$.

Proof: $\exists\, r \neq 0,\ r \in \mathbb{Z},\ \left|\sum_{S \in A} e^{\frac{2\pi i}{n} sr}\right| \geq \frac{|A|}{2}$.

(2) [4] $1 \leq m < n$, $m, n \in Z$, $a_{ij} \in \mathbb{N}$, where $1 \leq i \leq n,\ 1 \leq j \leq m$, $A_j = \sum_{i=1}^{n} |a_{ij}|$.

$\forall\, 1 \leq j \leq m,\ A_j \neq 0$. Proof: There exist integers $x_i\ (1 \leq i \leq n)$ that are

not all zero, and $|x_i| \leq \prod_{i=1}^{m} A_j^{\frac{1}{n-m}}$, $\sum_{i=1}^{n} a_{ij} x_i = 0$.

(2) [5] Proof: There exist $c > 0$, such that

$$\forall n \in N^+,\ \exists\, A \subseteq \{1, 2, 3 \ldots n\},\ |A| \geq n e^{-c\sqrt{\ln N}}$$

and there are no three terms in A that form an arithmetic sequence.

(4) [6] For $n \geq 2$, denote A_n as the set of factors of an n-degree polynomial with all coefficients belonging to $\{-1, 0, 1\}$. Let $C(n)$ be the largest coefficient of a polynomial with integer coefficients belonging to A_n.

Proof: For $\forall \varepsilon > 0$, $\exists\, k \in N^+$, such that

$$\forall n > k,\ 2^{n^{\frac{1}{2}-\varepsilon}} < C(n) < 2^n.$$